Overview

- 4 series of invariants are related via physical intuitions ("duality")

① instanton counting
$=$ equivariant Donaldson invariants of $\mathbb{R}^4$
or $=$ Hilbert series of the coordinate rings of quiver varieties

②. Gromov-Witten invariants of certain noncompact Calabi-Yau 3-folds (with lagrangian submfds)

③. open GW invariants of T^*S^3 (deformed conifold) (with lagrangian submanifolds)

④. Chern-Simons (or Jones-Witten) invariants of (links in) S^3

○ Their links via physical intuitions

1

$\updownarrow$ "geometric engineering"

2 (Katz–Klemm–Vafa hep-th/9609239)

and others ---

2

$\updownarrow$ "large N duality"

3 (Gopakumar–Vafa hep-th/9811131)

(Ooguri–Vafa hep-th/9912123)

3

$\updownarrow$ Chern-Simons theory as a string theory

4 Witten hep-th/9207094

based on · Chern–Simons perturbation theory

· 't Hooft expansion

All invariants (except for possibly open GW inv.) are mathematically rigorously defined, but their links are very mysterius for mathematicians (at least for me).

In most of cases invariants are computable.
So the identifications of invariants
are possible in rigorous ways.

For ① "instanton counting", we use the localization
(or Bott's, Lefschetz') formulas for the
equivariant cohomology (or K-theory).

$\rightsquigarrow$ invariants = sum over partitions

For ② GW invariants (of toric Calabi-Yau), (toric: type A)
we again use the localization formula.
We further compute certain Hodge integrals
over the moduli space of Riemann surfaces.

$\rightsquigarrow$ topological vertex
again invariants = sum over partitions

Rem. "geometric engineering" (connection ① & ②)
can be checked by computing
both invariants.

○ 1-parameter deformation

① instanton counting

replace S^1 by T^2 in equiv. cohomology

④ link invariants

Khovanov homology (more precisely its Poincaré polynomial

②, ③ $\rightsquigarrow$ <u>mysterious</u>

need to add one more parameter to the genus expansion parameter.

So far it is explained via "BPS states countings", the Poincaré polynomial of the cohomology of certain moduli spaces.

But a mathematical justification is <u>not</u> given so far, as moduli spaces are singular in general.

<u>Rem</u>. BPS counting (for a general CY 3-fold) gives an integrality conjecture of GW invariants (Gopakumar - Vafa hep-th/9809187)

○ example

②, ③ GW for CY 3-folds

conifold : $xy = uv$ in $\mathbb{C}^3$

— deformation $xy = uv + t$ t : cpx parameter

diffeomorphic to T^*S^3

— resolution

total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$

(= normal bundle of rigid $\mathbb{P}^1$ in CY 3fold)

$([z_0 : z_1], \xi, \zeta) \longmapsto x = \xi z_0,\ y = \zeta z_1$

$u = \zeta z_0,\ v = \xi z_1$

We consider (open) GW invariants of the resolved / deformed conifold

large N duality

Two GW invariants are equal !

More precise look on GW invariants

- resolved conifold X = total sp. of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$

  any nonconstant holomorphic map $C \longrightarrow X$ has the image in the zero section

$\overline{M}_{g,0}(\mathbb{P}^1, d)$ = moduli space of genus g, degree d stable maps to $\mathbb{P}^1$

$\mathcal{U}$ = universal family

$$\begin{array}{ccc} & \mathcal{U} & \\ \swarrow{\scriptstyle \pi} & & \searrow{\scriptstyle \mu} \\ \overline{M}_{g,0}(\mathbb{P}^1, d) & & \mathbb{P}^1 \end{array}$$

$$\int_{\overline{M}_{g,0}(\mathbb{P}^1, d)} c_{top}\left(R^1 \pi_* \mu^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))\right) =: C(g, d)$$

generating function (Faber – Pandharipande)

$$\sum_{g \geq 0} \sum_{d \geq 1} \mathfrak{z}^d g_s^{2g-2} C(g, d)$$

$$= \sum_{d \geq 1} \frac{1}{d} \left(2 \sin \frac{d g_s}{2}\right)^{-2} \mathfrak{z}^d$$

constant map $\overline{M}_{g,0}(X,0) \cong \overline{M}_{g,0} \times X$

$$\int_{\overline{M}_{g,0}(X,0)} c_{3g}(\mathbb{E}^* \boxtimes T_X)$$

$\mathbb{E}$: Hodge b'dle
$H^0(C, \omega_C)$

$$= \int_{\overline{M}_{g,0}} c_{g-1}(\mathbb{E})^3 = \frac{(1-2g) B_{2g} B_{2g-2}}{(2g-2)(2g)!}$$

③ open GW inv. of the deformed conifold

We must consider "open" strings
(Riemann surface with bdry)

$S^3 \supset L$: link
$\rightsquigarrow \quad T^*S^3 \supset C_L$: conormal b'dle

$f : (\Sigma, \partial\Sigma) \longrightarrow (T^*S^3, C_L)$ holo. map

Then $\displaystyle\sum_{g,h} (\# \text{ of such maps}) \times g_s^{2g-2}\, t^{h = \# \text{ of bdrys}}$

- No mathematically rigorous definition so far
- Via the large N duality the above closed GW inv. for the resolved conifold are equal to the open GW inv. with $L = \phi$. But I do not know how this should be understood if $L = \phi$.

④ Chern–Simons invariants for (M^3, L)
(to be explained in the next lecture)

③ $\leftrightarrow$ ④ "CS as a string theory"

$M = S^3$

i.e. $\boxed{\begin{array}{l} \text{open GW invariants for } (T^*S^3, C_L) \\ = SU(N)\text{-CS invariants for } (S^3, L) \end{array}}$

where $g_s = \frac{2\pi}{k+N}$, $t = \frac{2\pi i N}{k+N}$

k: level

<u>Note</u> N turns to be a variable.

As I explained, ③ is not rigorously defined yet. so this is not quite mathematically well formulated conjecture.

② ⟷ ③ Large N duality
(geometric transition)

Conj: $\forall L$: link $\subset S^3$
Ooguri
-Vafa
$\exists\ \widetilde{C}_L \subset X$: resolved conifold lagrangian

s.t. $\widetilde{C}_L \sim C_L$ at the end

open GW inv. of $(X, \widetilde{C}_L)$

$=$ open GW inv. of (T^*S^3, C_L)

$=$ CS inv. of (S^3, L)

(The construction of $\widetilde{C}_L$ is not explained.)

As I said open GW inv. is not rigorously defined.

In some examples of link L,
CS inv. fn (S^3, L) can be identified
with • closed GW inv. of X_L
toric CY

or
• relative GW inv. (as in the case
of topological vertex

Though I do not understand how X_L is
constructed.

Example when $L = \phi$,
$X_L = X_\phi$ = resolved conifold !

closed GW inv. of the resolved conifold
= CS inv. of S^3

① $U(1)$ - case

moduli space of $U(1)$-instantons on $\mathbb{R}^4$

$$= S^n \mathbb{R}^4 \quad \text{(symmetric product)}$$

(genuine $U(1)$-instanton ······ only trivial connection
the above is the moduli of singular instantons

$$T^2 \curvearrowright \mathbb{R}^4 \cong \mathbb{C}^2$$
$$\underset{\psi}{} (e^{\varepsilon_1}, e^{\varepsilon_2}) \qquad (x, y) \mapsto (e^{\varepsilon_1} x, e^{\varepsilon_2} y)$$

equivariant $U(1)$-Donaldson invariant

$$= \text{"} \int_{S^n \mathbb{R}^4} 1 \text{"}$$

This integration is WRONG in 2 points :
- $S^n \mathbb{R}^4$ is noncompact
- 1 is a function, not a top degree form

correct definition : formal application of the localization formula

$$= \frac{1}{\# S_n} \frac{1}{e(T_0 \mathbb{R}^{4n})} = \frac{1}{n!} \frac{1}{(\varepsilon_1 \varepsilon_2)^n}$$

generating series :

$$\sum_n \mathfrak{q}^n \int_{S^n\mathbb{R}^4} 1 = \exp\left(\frac{\mathfrak{q}}{\varepsilon_1\varepsilon_2}\right)$$

– K-theoretic equivariant U(1)-Donaldson invariant

integration in K-theory = index

or holo. Euler char.

$$\therefore \sum (-1)^i \operatorname{ch} H^i(S^n\mathbb{C}^2, \mathcal{O})$$

= character of the coordinate ring of $S^n\mathbb{C}^2$

$$= \operatorname{ch} \mathbb{C}[x_1, y_1, \cdots, x_n, y_n]^{S_n}$$

<u>Exercise</u> : generating function $= \exp\left(\dfrac{\mathfrak{q}^d}{(1-e^{\varepsilon_1 d})(1-e^{\varepsilon_2 d})\, d}\right)$

① = ② geometric engineering

$$\boxed{\begin{array}{l}\text{nonconstant map contribution of GW} \\ \quad = \text{K-theoretic equivariant} \\ \qquad \text{U(1)-Donaldson invariant} \\ \text{with } \varepsilon_1 = i g_s,\ \varepsilon_2 = -i g_s\end{array}}$$

(one parameter is dropped.)

◎ Combinatorial expression

localization formula (explained in more detail in later lectures)

$\mathrm{Hilb}^n \mathbb{C}^2 \longrightarrow S^n \mathbb{C}^2$ resolution of singularities T^2-equivariant

$(\mathrm{Hilb}^n \mathbb{C}^2)^{T^2}$ = fixed point set is parametrised by partitions of n

$$\rightsquigarrow \int_{S^n \mathbb{R}^4} 1 = \int_{\mathrm{Hilb}^n \mathbb{C}^2} 1$$

$$= \sum_{(\mathrm{Hilb}^n \mathbb{C}^2)^T \ni p_\lambda} \frac{1}{e(T_{p_\lambda} \mathrm{Hilb}^n \mathbb{C}^2)}$$

K-theory $\sum (-1)^i \mathrm{ch}\, H^i(S^n \mathbb{C}^2, \mathcal{O}) = \sum (-1)^i \mathrm{ch}\, H^i(\mathrm{Hilb}^n \mathbb{C}^2, \mathcal{O})$

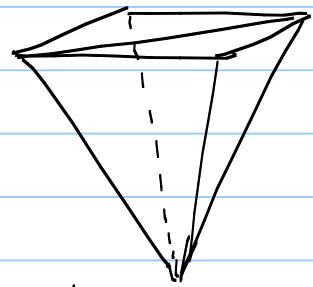
$$= \sum_{\lambda \vdash n} \frac{1}{\mathrm{ch}(\Lambda_{-1} T^*_{p_\lambda} \mathrm{Hilb}^n \mathbb{C}^2)}$$

In fact, there is purely combinatorial proof of $=$ via the Cauchy formula for Macdonald polynomials.

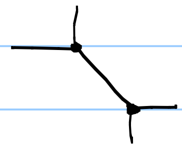
GW for resolved conifold

resolved conifold is a toric variety.

fan



dual graph



torus fixed point ---- 2 pts
(0 & ∞ in $\mathbb{P}^1$)

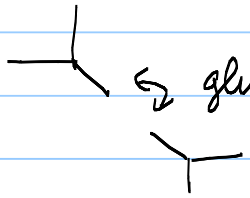
Toric varieties are obtained by patching $\mathbb{C}^3$'s
(toric n.b.ds around fix points)

Topological vertex computes the "local contribution"
of $\mathbb{C}^3$, and then gives the gluing formula

- original physical idea (Aganagic-Klemm-Marino-Vafa)
  hep-th/0305132
  open GW of $\mathbb{C}^3$ with lagrangian bdry

- mathematical version (Li-Liu-Liu-Zhou)
  math.AG/0408426
  relative (closed) GW

For the resolved conifold



gluing $\rightsquigarrow \sum_{\text{partition}}$

the same expression as in the localization formula in Hilb.